Modified Keller-Segel system and critical mass for the log interaction kernel

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Abstract

The Keller-Segel system has been widely validated as a model for cell movement under the combined effect of stochastic diffusion and a directed drift due to a chemoattractant emitted by the cells themselves (chemotaxis). In two dimensions, and in its simpler mathematical setting, it has been proved that the system admits a critical mass (below this mass there are global solutions, above the system blows-up). In higher dimensions this is not true and $L^{d/2}$ is the critical norm but only less precise statements are available.

Here, we study a variant for the diffusion law of the chemical potential in chemotaxis, based on a convolution operator or in other words on a fractional diffusion law. The aim of this paper is to enumerate the corresponding qualitative properties – global existence, blow up, stationary states – for the model with critical logarithmic kernel. In particular we show that this new system leads to a unified theory with critical mass in any dimension.

Key words. Chemotaxis, global existence, blow-up, fractional diffusion, critical mass.
AMS subject classifications. 35B60; 35Q80; 92C17; 92C50.

1 Introduction

In biology, chemotaxis refers to collective cell movements directed by their interaction through an attractive chemical potential, the *chemoattractant*. This aggregation tendency is counter-balanced by diffusion of cells due to their brownian motion. Chemotaxis models generally couple an equation for the movement of cells ($n(t,x)$ denotes the cell density) together with the chemical constituent which is produced by the cells themselves ($c(t,x)$ denotes the concentration of the chemoattractant).

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The mathematical question which arises in this class of models is to determine which contribution will dominate, either aggregation by directed chemoattraction or dispersion by stochastic diffusion. As we recall it below the answer depends highly upon the space dimension and makes the situation rather complex. Let us recall that in biology the experiments are carried out on a two dimensional dish and three dimensional effects are usually neglected. In astrophysics there is a similar model in three dimensions.

Here we introduce another form of the diffusion law for the chemical which is aimed at providing properties that are dimension invariant. Namely we consider the Modified Keller-Segel system (MKS in short) set on an open subset Ω of $\mathbb{R}^d$, and for simplicity it is either a regular bounded domain or the whole space $\mathbb{R}^d$,

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot \big(n \nabla c\big) & t > 0,\ x \in \Omega \subset \mathbb{R}^d, \\ c = K_d * n & t > 0,\ x \in \Omega \subset \mathbb{R}^d, \\ \dfrac{\partial n}{\partial \eta} - \chi n \dfrac{\partial c}{\partial \eta} = 0 & t > 0,\ x \in \partial\Omega, \\ n(t=0) = n_0 \geq 0, \end{cases} \tag{1}$$

where η denotes the outward unit normal to Ω when it is a bounded domain. The critical kernel K_d and mass $M_{\rm crit}$ are defined as

$$K_d(z) = -\frac{1}{d\pi} \log |z|, \qquad M_{\rm crit} = \frac{2d^2\pi}{\chi}. \tag{2}$$

Although the main interest is the better mathematical behavior of this system compared to the classical Keller-Segel system (see next section), there are several other motivations. Firstly we will see that even in one dimension, this system blows-up over a critical mass and this renders possible elementary numerical studies of this highly subtle phenomena (see the non self-similarity of the explosion in [18, 32], and as pointed out in [26, 32] the continuation after blow-up depends upon the regularization). Secondly there is a recent interest on fractional diffusions in biology because the molecules undergo specific interactions with the overall medium.

The outline of the paper is as follows. We begin with a short presentation of the classical Keller-Segel system and its main properties. Then we study the MKS for both the whole space (section 3) and a bounded domain (section 4). We show the existence of a critical mass, based on the free energy and the second moment of n.

2 The classical framework and motivation for a log kernel

In any dimension the classical Keller–Segel model for chemotaxis is (see [22, 21])

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot \big(n \nabla c\big) & t > 0,\ x \in \Omega, \\ -\Delta c = n & t > 0,\ x \in \Omega. \end{cases} \tag{3}$$

We would like first to emphasize the role of boundary conditions. For the cell density we impose as usual zero-flux boundary conditions,

$$\frac{\partial n}{\partial \eta} - \chi n \frac{\partial c}{\partial \eta} = 0 \quad t \geq 0\,,\ x \in \partial\Omega, \tag{4}$$

whereas we distinguish between Neumann and Dirichlet boundary conditions (BC in short) for the chemical concentration,

$$\frac{\partial c}{\partial \eta} = 0 \quad \text{or} \quad c = 0 \quad t \geq 0\,,\ x \in \partial\Omega. \tag{5}$$

Notice that (4) and $n(t=0) = n_0 \geq 0$ guarantees nonnegative solutions $n(t,x) \geq 0$ and mass conservation

$$\int_\Omega n(t,x)dx = \int_\Omega n(t=0,x)dx := M. \tag{6}$$

Remark 2.1 *In case of Neumann BC, equation for c has to be replaced with*

$$-\Delta c = n - \langle n \rangle_\Omega.$$

Therefore c is not interpreted as a concentration, but as a deviation to meanvalue.

Remark 2.2 *We made the choice of a very simple description of the classical chemotaxis system. Note that the equation for chemical may be replaced with the more physical laws* $-\Delta c + c = n$ *or* $\partial_t c - \Delta c = n$. *But more elaborate models are used in practice, see [27, 6].*

In the particular case where $\Omega = \mathbb{R}^2$, the Poisson equation $-\Delta c = n$ becomes

$$c(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| n(t,y)dy. \tag{7}$$

The interesting feature of the model (3) is that solutions may become unbounded in finite time. We call *blow-up in finite time* such a behavior. Since the 70's and the seminal papers by Keller and Segel [22, 23], a great effort has been made in this direction, and these equations are now quite well understood. We propose to review the main results in this area. First of all, let us remark that the behavior of the system (3) highly depends on the dimension. In dimension $d = 1$ no solutions blow up [19] whereas in dimension $d \geq 3$ there exist blowing up solutions for any positive mass but this is not the correctly scaled variable in this case. The precise condition for existence involves small $L^{d/2}$ norm of n_0 and has been derived in [12] for the parabolic-elliptic and in [11] for the parabolic-parabolic KS systems. Always for $d > 2$, it is not known if blow-up follows from large $L^{d/2}$ norm of n_0 (only results involving a stronger norm that scales similarly are known). In the critical dimension $d = 2$ there is a threshold: solutions are global in time for small mass, and blow up for large mass.

This threshold phenomena is quite simple in the whole space $\mathbb{R}^2$, and is summarized in the following theorem [13, 4].

Theorem 2.3 (Critical mass for the KS model in $\mathbb{R}^2$)
If Ω is the whole space $\mathbb{R}^2$ and $n_0\big(|\log n_0| + (1+|x|^2)\big) \in L^1$, then solutions are global in time if $\chi M < 8\pi$, or blow up in finite time if $\chi M > 8\pi$.

On a bounded domain the analysis is more difficult because boundary effects play an important role. Concerning Dirichlet conditions, the situation is similar to the whole space: solutions are global in time if $\chi M < 8\pi$, and blow up in finite time if $\chi M > 8\pi$. On the other hand, there are several threshold values in the case of Neumann boundary conditions: if Ω is regular then solutions are global if $\chi M < 4\pi$, and may blow up above this threshold, either on the boundary or inside the domain. If Ω is piecewise $\mathcal{C}^2$ and Θ denotes the smallest angle then solutions are global if $\chi M < 4\Theta$, and may blow up above this threshold [20, 8]. On a disc, the situation is more clear and has been analyzed in [3] and it turns out that, again, boundary conditions on c play an important role.

Our goal in this paper is to study the Modified Keller-Segel system (1)–(2) and to generalize the phenomenon of blow-up to all dimensions. We start from the following remark: the dimension $d = 2$ is critical because the weight of the interacting kernel $K(z) = -\frac{1}{2\pi}\log|z|$ is itself critical. If we replace the equation for the chemical potential $-\Delta c = n$ by $c = K_d * n$, the corresponding behavior depends slightly on the dimension, namely through the threshold value. There is some hidden subtlety in this formulation in the case of a bounded domain Ω. For the convolution product $K_d * n$ to be well defined, it is necessary to extend n outside Ω. We distinguish as well two natural ways: the extension by 0, which corresponds in a certain sense to Dirichlet boundary conditions; and the extension by meanvalue which corresponds to Neumann boundary conditions. The main difference is that $n = constant$ is a stationary solution only for Neumann BC for KS model and for the extension by meanvalue for the MKS model.

3 MKS system in the whole space

There is a free energy naturally associated to the system (1), namely the Lyapunov functional [16, 1]

$$\mathcal{F}(n) = \int n\log n - \frac{\chi}{2}\int nc = \int n\log n - \frac{\chi}{2}\int n\ K_d * n, \tag{8}$$

satisfies $t \to \mathcal{F}\big(n(t)\big)$ is decreasing, more precisely

$$\frac{d}{dt}\mathcal{F}\big(n(t)\big) = -\int n|\nabla\log n - \chi\nabla c|^2 \le 0. \tag{9}$$

From the *logarithmic Hardy-Littlewood-Sobolev inequality* [9] it comes that the functional (8) is bounded from below for small mass.

Theorem 3.1 (Logarithmic Hardy-Littlewood-Sobolev inequality)
Assume f is a nonnegative function $\mathbb{R}^d \to \mathbb{R}$ with total mass M and $f(x)\log(1+|x|^2)$ integrable, then

$$-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\log|x-y|f(y)\,dx\,dy \le \frac{M}{d}\int_{\mathbb{R}^d} f\log f\,dx + C(d,M).$$

As a consequence,

$$\begin{aligned}\mathcal{F}(n) &= \int n\log n + \frac{\chi}{2d\pi}\int n\ \log * n\\ &\ge \Big(1-\frac{\chi M}{2d^2\pi}\Big)\int n\log n + C\\ &= \Big(1-\frac{M}{M_{\rm crit}}\Big)\int n\log n + C.\end{aligned}$$

We deduce from this lower bound that the cell density n is uniformly equi-integrable as soon as $M < M_{\rm crit}$.

Theorem 3.2 (Critical mass for a log kernel)
Let Ω be the whole space $\mathbb{R}^d$ and $n_0\big(|\log n_0| + 1 + |x|^2\big) \in L^1$. Assume $M > M_{\rm crit}$ then solutions to (1) blow-up in finite time. If $M < M_{\rm crit}$, then the system (1) has a global weak solution and L^p regularity is propagated; if additionally $n_0 \in L^p$ for some $p > d > 1$ then $n(t,x) \in L^\infty\big((\alpha,T)\times\mathbb{R}^d\big)$ for all $T > \alpha > 0$.

One can check that $p > d/2$ is enough for the regularizing effect in L^∞ (see [11]) but the argument is based on iterations that make it longer to present and we prefer to skip this technical issue.

The end of this section is devoted to the proof of these results.

Blow-up and weak solutions. To prove that solutions blow up in finite time we show, following [28], that the second momentum of n cannot remain positive for all time. It

relies on the following computation in the case $d \geq 2$,

$$\begin{aligned}
\frac{d}{dt}\int \frac{1}{2}|x|^2 n(x,t)\,dx &= \int \frac{|x|^2}{2}\frac{\partial}{\partial t} n\,dx \\
&= \int \frac{1}{2}|x|^2 \nabla\cdot(\nabla n - \chi n \nabla c)\,dx \\
&= -\int x\cdot(\nabla n - \chi n\nabla c)\,dx \\
&= \int (\nabla\cdot x) n\,dx - \frac{\chi}{d\pi}\iint n(x)\frac{x\cdot(x-y)}{|x-y|^2} n(y)\,dy\,dx \\
&= M\Big(d - \frac{\chi}{2d\pi}M\Big) \\
&= dM\Big(1 - \frac{M}{M_{\rm crit}}\Big). \qquad (10)
\end{aligned}$$

For $d = 1$ the computation is slightly different but the final result is the same. We have $\nabla c = -Hn$, where H denotes the Hilbert transform [14]. We obtain therefore

$$\begin{aligned}
\frac{d}{dt}\int \frac{1}{2}|x|^2 n(x,t)\,dx &= dM - \frac{\chi}{d\pi}\int n(x)\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{x}{x-y} n(y)dydx \\
&= dM - \frac{\chi}{2d\pi}\lim_{\varepsilon\to 0}\iint_{|x-y|>\varepsilon} n(x)n(y)dxdy \\
&= dM\Big(1 - \frac{M}{M_{\rm crit}}\Big). \qquad (11)
\end{aligned}$$

For $M > M_{\rm crit}$, it proves that some singularity occurs that prevents this computation to be possible (otherwise there is a contradiction between the positivity of the second moment and its negative decay). The singularity can be analyzed (making rigorous by regularization the above calculation) and it is proved in [13, 4] that ∇c cannot remain bounded.

For $M \leq M_{\rm crit}$, it gives a local in time control of the second moment which is important because it allows to use (see again [13, 4] for instance) the standard inequality for $n(x) \geq 0$,

$$\int n(x)\log_- n(x)dx \leq C\Big(\int n(x)dx, \int |x|^2 n(x)dx\Big).$$

This provides a critical control, specific to the case $\Omega = \mathbb{R}^d$ which is enough, with the $L\log L$ bound, to prove the existence of weak solutions, see [13, 4].

Stationary states. One can look for steady states which are minimizers of the energy functional $\mathcal{F}$ (see section 4.2). In the case of the critical mass $M = M_{\rm crit}$ and the whole

space $\mathbb{R}^d$ we know precisely the minimizers of $\mathcal{F}$ [9]: they are given by the conformal images of the function

$$h(x) = |\mathbb{S}^d|^{-1} \left(\frac{2}{1+|x|^2} \right)^d .$$

Propagation of L^p bounds. As soon as equi-integrability is gained, the remaining work is usually to propagate L^p regularity for the cell density. This work was initiated by Jäger and Luckhaus [21] who pointed out that the Gagliardo–Nirenberg–Sobolev (GNS) inequality plays a key role within these estimates. Applying their method to model (1) we have to distinguish between dimension $d = 1$, $d = 2$ and $d \geq 3$. The case $d = 2$ has already been well studied, because it is the classical KS model [21, 16, 2, 12, 7]. Nevertheless we explain briefly the strategy, based on the following computation,

$$\frac{d}{dt} \int n^p = -4 \frac{p-1}{p} \int |\nabla n^{p/2}|^2 + \chi(p-1) \int n^{p+1}. \tag{12}$$

We cannot apply a Gronwall lemma here, but the GNS inequality [15, 30] enables to compare the two opposite terms of the right-hand-side, namely

$$\int n^{p+1} \leq C(p) \|\nabla n^{p/2}\|_2^2 \int n. \tag{13}$$

Closing this computation requires a suboptimal mass condition which depends on p. In order to circumvent this difficulty, we still follow [21] and we can replace n by $(n-k)_+$. We obtain finally for $p > 1$:

$$\begin{aligned} \frac{d}{dt} \int (n-k)_+^p \leq \left(-4 \frac{p-1}{p} + C(p) \chi (p-1) \int (n-k)_+ \right) \|\nabla (n-k)_+^{p/2}\|_2^2 \\ + O\left(\int (n-k)_+^p \right). \end{aligned} \tag{14}$$

This proves L^p regularity for all p because the term $\int (n-k)_+$ is uniformly small for large k, just using the upper bound

$$\int (n-k)_+ \leq \int_{n \geq k} n \frac{\log n}{\log k} \leq \int n \frac{\log_+ n}{\log k} \leq \frac{C}{\log k}.$$

In the following we would like to apply the same strategy to the cases $d = 1$ and $d \geq 3$. Although computations are more complex, they are precisely similar to $d = 2$.

For $\mathbf{d \geq 3}$ the time derivative of the L^p norm becomes

$$\frac{d}{dt} \int n^p = -4 \frac{p-1}{p} \int |\nabla n^{p/2}|^2 + \chi(p-1) \int n^p \big(- \Delta c \big).$$

We focus on the last term,

$$\int n^p\big(-\Delta c\big) = \int n^p\Big(\big(-\Delta K_d\big) * n\Big).$$

In the sense of distributions we have $-\Delta K_d(z) = \dfrac{1}{d\pi}\dfrac{d-2}{|z|^2}$. In addition we propose to apply the Hardy-Littlewood-Sobolev inequality [24] together with a special case of the GNS inequality [15, 30],

$$\begin{aligned}
\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)|x-y|^{-\lambda}g(y)\,dx\,dy\right| &\le C(d,\lambda,q)\|f\|_q\|g\|_r\ ,\\
q,r>1\ ,\ 0<\lambda<d\ ,\quad &\frac{1}{q}+\frac{\lambda}{d}+\frac{1}{r}=2\ ;\\
\int n^{p+2/d} \le C(p)\|\nabla n^{p/2}\|_2^2\left(\int n\right)^{2/d}&. \qquad (15)
\end{aligned}$$

Finally it comes using interpolation inequality that

$$\begin{aligned}
\int n^p\big(-\Delta c\big) &= \frac{d-2}{d\pi}\iint n^p(x)|x-y|^{-2}n(y)\,dx\,dy,\\
&\le C\|n^p\|_q\|n\|_r,\\
&\le C\|n^p\|_q\|n\|_{p+2/d}^{\theta}\|n\|_1^{1-\theta},\\
&\le C\left(\int n^{p+2/d}\right)^{\frac{1}{q}+\frac{\theta}{p+2/d}}\|n\|_1^{1-\theta},
\end{aligned}$$

where the numerology q, r, θ satisfies

$$pq = p+\frac{2}{d}\ ,\quad \frac{1}{q'}+\frac{1}{r'}=\frac{2}{d}\ ,\quad \theta=\frac{2}{d}\ ,\quad \frac{1}{q}+\frac{\theta}{p+2/d}=1.$$

We are now reduced to the case where GNS inequality (15) can be used. We can redo the computations with $(n-k)_+$ instead of n to get an estimation corresponding to (14) in case $d=2$ and the conclusion is similar.

For $\mathbf{d=1}$ the term $-\Delta K_d$ becomes more complex. In the sense of distributions it is

$$\langle -\Delta K_d, \phi\rangle = -\frac{1}{\pi}\ \text{p.v.}\frac{1}{x}\Big(\frac{d\phi}{dx}\Big).$$

In other words, Hf denoting the Hilbert transform of f [14], we have $\nabla c = -Hn$. The Hilbert transform is strong (p,p) for each $p>1$ (M. Riesz), that is $\|Hf\|_p \le C(p)\|f\|_p$.

We can rebuild the same argumentation as previously,

$$\begin{aligned}
\frac{d}{dt}\int n^p &= -4\frac{p-1}{p}\int |\nabla n^{p/2}|^2 + \chi(p-1)\int \nabla n^p\cdot\nabla c.\\
\int \nabla n^p\cdot\nabla c &\le \|\nabla n^{p/2}\|_2\left(\int n^p|Hn|^2\right)^{1/2},\\
&\le \|\nabla n^{p/2}\|_2\left(\int n^{pq}\right)^{\frac{1}{2q}}\|Hn\|_{2q'},\\
&\le C\|\nabla n^{p/2}\|_2\left(\int n^{p+2}\right)^{1/2},
\end{aligned}$$

with the following definition of exponents: $pq = p+2 = 2q'$. We can apply the inequality (15) once again and replace n by $(n-k)_+$. Then it is just the same as in dimension 2.

L^∞ bound. The regularizing effect in L^∞ can be obtained using the heat kernel $G(t,x)$. We have

$$n(t) = n_0 * G(t) - \int_0^t \nabla G(t-s) * (n\nabla c)(s)ds.$$

Since we know that $n \in L^\infty\big((0,T);L^p(\mathbb{R}^d)\big)$ (and $p > d'$), we also have $\nabla c = -\frac{1}{d\pi}\frac{x}{|x|^2} * n$ belongs to $L^\infty\big((0,T)\times\mathbb{R}^d\big)$. Therefore, we arrive at

$$\begin{aligned}
\|n(t)\|_{L^\infty} &\le \frac{C}{t^{d/2}}\|n_0\|_{L^1} + C\int_0^t \|\nabla G(t-s) * n(s)\|_{L^\infty}ds\\
&\le \frac{C}{t^{d/2}}\|n_0\|_{L^1} + C\|n\|_{L^\infty\left((0,T);L^p(\mathbb{R}^d)\right)}\int_0^t \|\nabla G(s)\|_{L^{p'}}.
\end{aligned}$$

And we conclude the bound since $\|\nabla G(s)\|_{L^{p'}} = Cs^{-\frac{1}{2}(1+\frac{d}{p})}$ and $\frac{1}{2}(1+\frac{d}{p}) < 1$ for $p > d$.

4 MKS system in a bounded domain.

From now on, we consider the model (1) set on a bounded domain $\Omega \subset \mathbb{R}^d$. First of all, we note that the model is homogeneous with respect to Ω. More precisely, considering $\lambda\Omega$ as the new domain and letting $\tilde{n}(t,x) = \lambda^d n(t,\lambda x)$ (the total mass is unchanged), then we recover the equation in Ω with a λ^2 time scaling.

The definition of c by a convolution forces us to extend n outside the domain Ω. We study separately two natural ways to perform this extension: by zero (section 4.1) and by meanvalue (section 4.3).

4.1 The cell density is extended by zero. Evolution

As mentioned earlier, this situation corresponds to Dirichlet BC on c, and is qualitatively similar to the whole space model. Starting from the main equation we get that this system is equipped with an energy functional,

$$\mathcal{F}(n) = \int_\Omega n \log n - \frac{\chi}{2} \int_\Omega nc \ , \quad c = K_d * n, \tag{16}$$

which satisfies, as in the case of the full space,

$$\frac{d}{dt}\mathcal{F}\big(n(t)\big) = -\int_\Omega n|\nabla \log n - \chi \nabla c|^2 \leq 0. \tag{17}$$

Because we can replace Ω by $\mathbb{R}^d$ in these formula, global existence under the critical mass follows the same structure as section 3: equi-integrability then L^p bounds. Moreover computations are exactly the same because n is well defined in whole $\mathbb{R}^d$.

Blow up. Again we follow [28] and the previous computation of the second momentum of n,

$$J(t) = \int_\Omega \frac{1}{2}|x|^2 n(x,t)\, dx. \tag{18}$$

Its time derivative satisfies now (see above (10) and (11))

$$\begin{aligned} \frac{d}{dt}J &= -\int_\Omega x \cdot (\nabla n - \chi n \nabla c)\, dx \\ &= -\int_{\partial\Omega} (x\cdot\eta) n + \int_\Omega (\nabla \cdot x) n\, dx + \chi \int_\Omega x \cdot n \nabla c\, dx \\ &= -\int_{\partial\Omega} (x\cdot\eta) n + dM - \frac{\chi}{2d\pi} M^2, \end{aligned} \tag{19}$$

where η still denotes the outside normal unit vector on the boundary. We may choose the origin to be the center of the star-shaped domain Ω, ensuring that the first term in (19) is negative. Also the condition $\chi M > 2\pi d^2$ implies that $\frac{d}{dt}J \leq -\varepsilon < 0$. But J should remain positive and this contradicts the global existence of solutions.

We summarize these properties in the following theorem.

Theorem 4.1 (Critical mass; extension by zero)
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $n_0 \in L\log L$. Assume $M < M_{\rm crit}$, then the system (1) has a global weak solution. On the contrary assume Ω is a star-shaped domain and $M > M_{\rm crit}$, then solutions to (1) blow-up in finite time.

Remark 4.2 *We may ask whether the assumption of a star-shaped domain is necessary or not. Consider the system (1) set on an annulus $a < r < b$ together with radial symmetry. One cannot expect any concentration point of cells (it would break the symmetry).*

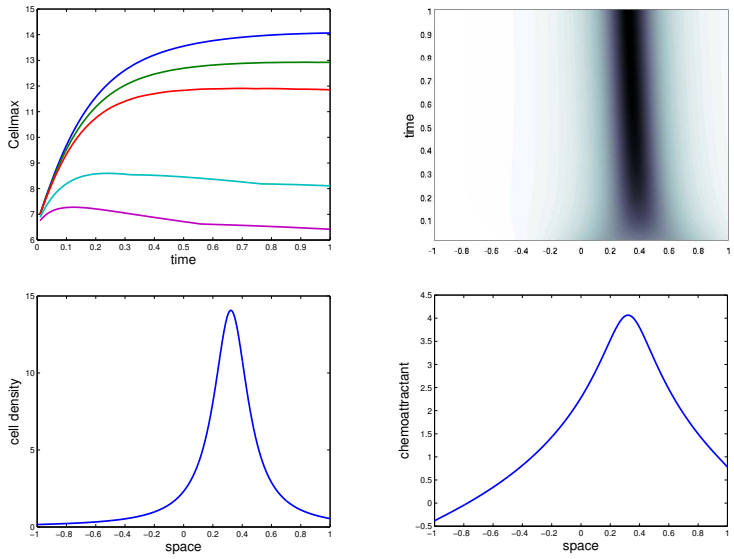

Figure 1: *(extension by zero)* In dimension 1, initially a gaussian with total mass $M = 1.8\pi < M_{\text{crit}}$ and chemosensitivity $\chi = 1$. (*up left*) The maximum value of the cell density is plotted for decreasing values of space step: $dx = [.001, .005, 0.01, 0.05, 0.1]$. (*up right*) The evolution of cell density corresponding to $dx = .001$. (*bottom*) Respectively the cell density and the chemical potential at the final time with $dx = .001$. See also figure 2.

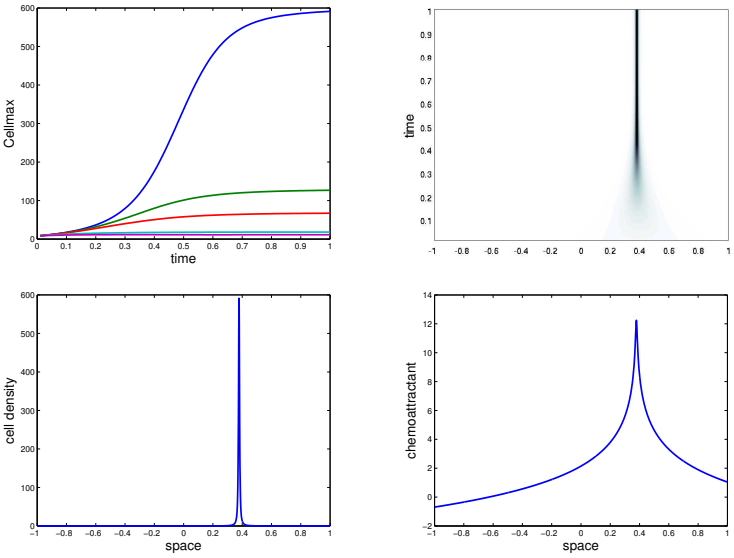

Figure 2: *(extension by zero)* Same as figure 1 with $M = 2.2\pi > M_{\text{crit}}$.

Numerics. We have performed numerical simulations for the system with extension by zero in dimension $d = 1$. The first point is to recover the critical mass from numerical experiments. Compared to the classical KS model there is an additional difficulty contained in the convolution term $c = K * n$. When it is discretized, the kernel K is truncated relatively to the space step ε, namely $K^\varepsilon(z) = -\frac{1}{\pi}\log\varepsilon$ if $|z| < \varepsilon$. We have plotted in figures 1 and 2 the maxima of cell density for decreasing values of ε together with the solution corresponding to the more accurate simulation, for $M < M_{\text{crit}}$ and $M > M_{\text{crit}}$ respectively. We note that these maxima highly depend on the space step. Consequently the notion of blow–up is difficult to track numerically in higher dimension. This illustrates one of the goals of the present model to also create a singularity in dimension 1.

As stated in the above theorem, the critical mass in this model is the same as in the whole space. The most remarkable property of the extension by zero is that constant distributions are not steady states because they do not satisfy the boundary conditions. Therefore it is of interest to study the possible steady distributions.

4.2 The cell density is extended by zero. Stationary states

From (17), we can deduce that the steady states are exactly the functions n which satisfies $\nabla \log n = \chi \nabla c$ or $\log n = \chi c + \mu$, where μ is determined by mass conservation [10]. So we end up with the following equations for steady states:

$$n = M\frac{e^{\chi c}}{\int_\Omega e^{\chi c}}\ , \quad c = K_d * n. \tag{20}$$

This section will be continued by a proof of the following

Theorem 4.3 (Steady states; extension by zero)
For $M < M_{\text{crit}}$ there is at least one solution to (20) with $c \in L^\infty$ and it is a minimizer for the energy functional $\mathcal{F}(n)$.

We are not aware of a uniqueness result for this problem, e.g. using convexity along proper paths as in [25].

For proving the above theorem, we begin with the main estimate we get from logarithmic Hardy-Littlewood-Sobolev inequality (theorem 3.1),

$$\mathcal{F}(n) \geq C + \Big(1 - \frac{M}{M_{\text{crit}}}\Big)\int_\Omega n\log n \geq C - \Big(1 - \frac{M}{M_{\text{crit}}}\Big)e^{-1}|\Omega|.$$

In other words $\mathcal{F}$ is bounded from below, because $x\log x$ is bounded from below. In the following, we let $K = K_d$. Let $\{n_k\}$ be a minimizing sequence. From the above inequality, we deduce that $\int_\Omega n_k\big(\log n_k\big)_+$ is bounded. Therefore the family $\{n_k\}$ is equi-integrable

and up to a subsequence we have $n_k \rightharpoonup n$ weakly in L^1. By convexity of the function $x\big(\log x\big)_+$, we can deduce that $n \in L\log L$ and more precisely

$$\int_\Omega n\big(\log n\big)_+ \leq \liminf \int_\Omega n_k\big(\log n_k\big)_+. \tag{21}$$

Using the duality inequality $ts \leq s\log s - s + e^t$, we show an L^∞ estimate for $c_k := K * n_k$. Choose the parameter $\alpha < \pi d^2$ and compute

$$\begin{aligned} c_k(x) &= \int_\Omega K(x-y)n_k(y)\ dy \\ &\leq \int_\Omega \frac{n_k}{\alpha}\log\frac{n_k}{\alpha} + \int_\Omega e^{\alpha K(x-y)}\ dy \\ &\leq -\frac{\log\alpha}{\alpha}M + \frac{1}{\alpha}\int_\Omega n_k\log n_k + \int_\Omega |x-y|^{-\frac{\alpha}{d\pi}}\ dy \\ &\leq C. \end{aligned}$$

Thus $\{c_k\}$ is bounded in $L^\infty(\Omega)$. Now for any test function $\phi \in L\log L(\Omega)$, $\check{K} * \phi$ is bounded and thus

$$\int_\Omega c_k\phi = \int_\Omega (K*n_k)\phi = \int_\Omega n_k(\check{K}*\phi) \longrightarrow \int_\Omega n(\check{K}*\phi) = \int_\Omega (K*n)\phi = \int_\Omega c\phi.$$

This shows that $c_k \rightharpoonup c = K * n$, weakly in L^q for all $q < \infty$.

In the following step we plan to deduce some compactness of $\{c_k\}$, say in L^1. First we remark that ∇K is integrable if $d \geq 2$, so that

$$\begin{cases} \nabla c_k = -Hn_k\ , & d = 1, \\ \nabla c_k = \nabla K * n_k\ , & d \geq 2, \end{cases}$$

where H is the Hilbert transform. On one hand, because $\int_\Omega n_k\big(\log n_k\big)_+$ is bounded we now that ∇c_k is bounded in L^1 in the case $d = 1$ [31, 5]. On the other hand ∇c_k is obviously bounded in L^1 for $d > 1$. Consequently $\{c_k\}$ is bounded in $W^{1,1}$ which is compactly embedded in L^1 for all dimensions. Up to a subsequence we have $c_k \to c$ strongly in L^1. In addition we can assume that $c_k \to c$ *a.e.* We invoke Egorov's theorem together with equi-integrability of $\{n_k\}$ and L^∞ bound on $\{c_k\}$ to conclude that

$$\int_\Omega n_k c_k \longrightarrow \int_\Omega nc.$$

Combining this with (21) we get finally that

$$\mathcal{F}(n) = \int_\Omega n\log n - \frac{\chi}{2}\int_\Omega nc \leq \liminf \mathcal{F}(n_k) = \inf \mathcal{F}.$$

This proves that n is a minimizer for $\mathcal{F}$.

Remark 4.4 *As $M \to M_{\rm crit}$ one can check that, along subseqences, the corresponding behavior occurs, for some $x^* \in \overline{\Omega}$,*

$$n \to M_{\rm crit}\delta(x-x^*) \ \ \textit{(weak sense of measures)},$$

$$c \to -\frac{2d}{\chi}\log|x-x^*| \ \ \textit{(in } L^p,\ 1\le p<\infty\textit{)}, \qquad \nabla c \to -\frac{2d}{\chi}\frac{x-x^*}{|x-x^*|^2} \ \ \textit{(in } L^p,\ 1\le p<d\textit{)}.$$

Indeed, since $\nabla n = \chi n \nabla c$, after testing against x and integration by parts, we obtain $\int_{\partial\Omega}(x\cdot\eta)n(x)\to 0$, in the case of a star-shaped domain Ω. But the lower bound on c, and (20) show that this can occur only when $\int_\Omega e^{\chi c}\to\infty$, and thus $\int_\Omega n\log n\to\infty$. In order to go further and prove there is a single concentration point as stated above, more elaborate but standard arguments that we do not copy, are needed along the lines of [29, 17].

4.3 Extension by mean value

The main drawback of the previous extension by zero is that constant densities are not steady states of the problem. To overcome this trouble we propose to extend n outside the domain by its meanvalue $\langle n\rangle_\Omega$. For this purpose we have to redefine the chemical potentiel by

$$c = K_d * \Big(n - \langle n\rangle_\Omega\Big), \qquad n = \langle n\rangle_\Omega \ \text{ in } \ \Omega^C, \tag{22}$$

which is the exact analogue of the Neumann BC for the KS model [21]. For convenience we define $\bar n = n - \langle n\rangle_\Omega$ so that $c = K_d * \bar n$. The decreasing free energy related to this system is the following

$$\mathcal{F}(n) = \int_\Omega n\log n - \frac{\chi}{2}\int_\Omega \bar n c, \qquad \bar n = n - \langle n\rangle_\Omega, \quad c = K_d * \bar n. \tag{23}$$

We are now ready to state our global existence or blow up result as follows

Theorem 4.5 (Critical mass; extension by meanvalue)
Let $\Omega\subset\mathbb{R}^d$ be a bounded domain and $n_0 \in L\log L$. Assume $M < M_{\rm crit}$, then the system (1) with $c = K_d*\big(n-\langle n\rangle_\Omega\big)$ has a global weak solution. On the contrary assume Ω is a star-shaped domain and $d\ge 2$; if $M > M_{\rm crit}$ and the initial second momentum $J(0)$ is small enough, then solutions to (1) blow-up in finite time.

Global existence. There are some minor technical changes between this section and the previous one. The key idea is that we can modify $\bar n$ into the quadratic term in (23) up to some constant, to recover the extension by 0. We define the piecewise constant function $\zeta = \langle n\rangle_\Omega \mathbf{1}_\Omega$, such that $\bar n + \zeta$ is the extension by zero and

$$\int_\Omega n\log n - \frac{\chi}{2}\int_\Omega \big(\bar n+\zeta\big)K_d*\big(\bar n+\zeta\big) \ \ = \ \ \mathcal{F}(n) + \text{bounded terms}.$$

The remaining terms are bounded because the kernel K_d is locally integrable. Keeping in mind this preliminary remark, the rest of the proof for existence is straightforward.

Remark 4.6 *Whereas in the classical KS model the critical masses differs between Dirichlet and Neumann BC (see section 2), in this new model they are the same. It seems surprising that the boundary curvature plays no effect on the mass threshold if the cell density is extended by its meanvalue. The reason is that we skip the boundary singularity of the Poisson kernel for a bounded domain in the logarithmic kernel* $K_d(z) = -\frac{1}{d\pi}\log|z|$.

Blow up. As in the previous sections we perform computations on the second momentum of n in the case $d \geq 2$ and a star-shaped domain $\Omega \subset \mathbb{R}^d$.

$$\begin{aligned}
\frac{d}{dt}J &= -\int_{\partial\Omega}(x\cdot\eta)n + dM - \frac{\chi}{2d\pi}M^2 + \frac{\chi}{d\pi}\langle n\rangle_\Omega \int_\Omega\int_\Omega n(x)\frac{x\cdot(x-y)}{|x-y|^2}\,dy\,dx \\
&\leq dM - \frac{\chi}{2d\pi}M^2 + \frac{\chi}{d\pi}\langle n\rangle_\Omega \int_\Omega\int_\Omega |x|n(x)|x-y|^{-1}\,dy\,dx \\
&\leq dM - \frac{\chi}{2d\pi}M^2 + \mathcal{C}\frac{\chi M}{d\pi|\Omega|}\int_\Omega |x|n(x)\,dx \\
&\leq dM - \frac{\chi}{2d\pi}M^2 + \mathcal{C}\frac{\chi M}{d\pi|\Omega|}\sqrt{MJ}.
\end{aligned}$$

We can be more accurate on the constant $\mathcal{C}$, which is

$$\mathcal{C} = \sup_{x\in\Omega}\int_\Omega |x-y|^{-1}\,dy \leq \frac{d}{d-1}\left(\frac{1}{d}|\mathbb{S}^{d-1}||\Omega|^{d-1}\right)^{\frac{1}{d}}.$$

which can be obtained by dividing the integral into $|x-y| \leq \lambda$ and $|x-y| > \lambda$ and optimizing the result with respect to λ. Finally we obtain that

$$\frac{d}{dt}J \leq dM - \frac{\chi}{2d\pi}M^2 + \frac{\chi}{\pi(d-1)}M^{\frac{3}{2}}\frac{\sqrt{J}}{R}. \tag{24}$$

with $|B(0,R)| = |\Omega|$. Consequently if $J(0)$ is small enough such that the right-hand-side of (24) is negative, then solutions blow up in finite time.

Remark 4.7 $\frac{\sqrt{J}}{R}$ *is homogeneous with respect to dilatations of* Ω *and this fact justifies the above calculations. Also one could not expect the blow up of solutions for large initial second momentums; for example initial constant distribution. So it is necessary to have an upper bound for* $J(0)$ *to ensure blowing up of solutions.*

Stationary states. Obviously constant distributions of cells are steady states in this case. However unicity is not clear. Furthermore in dimension $d = 2$ with radial symmetry and the classical chemical potentiel $-\Delta c = n - \langle n\rangle$, the numerical computation of steady states are easier. Using a shooting method we can observe a non–trivial steady state appearing for $M > M_{\text{crit}}$ (*unpublished results*).

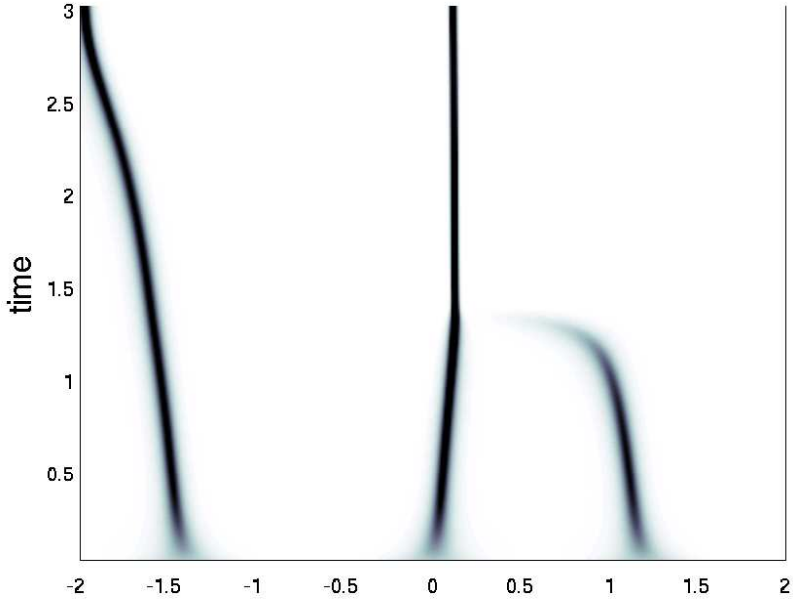

Figure 3: *(extension by meanvalue)* Initially 3 gaussian–like peaks with total mass $M = 7\pi > 3M_{\text{crit}}$ and chemosensitivity $\chi = 1$.

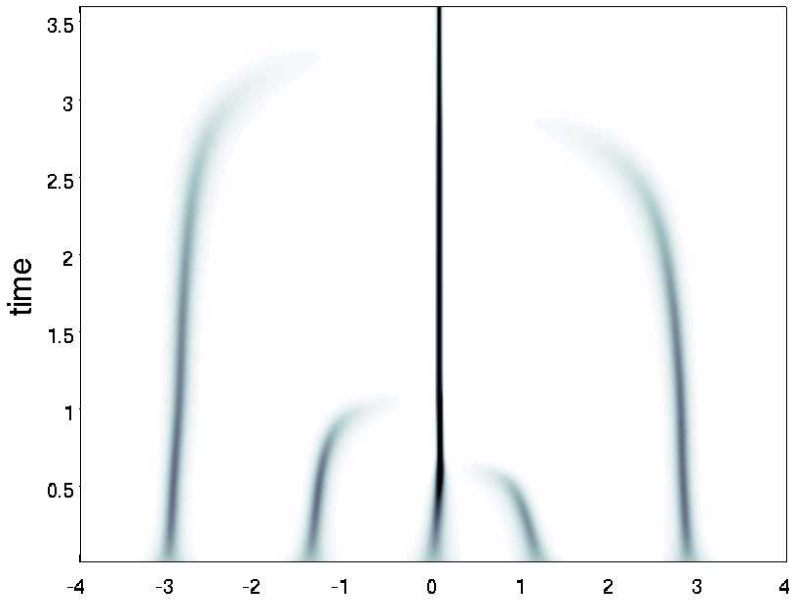

Figure 4: *(extension by meanvalue)* Initially 5 gaussian–like peaks with total mass $M = 12\pi > 5M_{\text{crit}}$ and chemosensitivity $\chi = 1$.

Numerics. We have performed simulations on a bounded domain with extension by meanvalue, in dimension $d = 1$. We have briefly tackled the problem of interaction between several peaks and the boundary. As it is the case in dimension two, the boundary induces an attractive effect on peaks (Fig. 3). Furthermore, if we start initially from several gaussian–like peaks which are sufficiently far from each other, then they start to aggregate cells around independently, and finally they attract each other (Fig. 4).

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